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ODD-DIMENSIONAL WIEDERSEHEN MANIFOLDS ARE SPHERES

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Dedicated to the author's teacher Professor Buchin Su

Let M be a connected, simply connected, compact Riemannian n-manifold without boundary, $n \geqslant 2$, such that for any $m \in M$, the cut locus of m in M is a single point. It is known that M is diffeomorphic to the n-sphere S^n. (This fact is not used in the present paper.) Moreover, every geodesic returns to its beginning point and is smoothly closed. Following Green [2], we call M a *wiedersehen n-manifold.*

It is easily seen that in M, all closed geodesics are of the same length, say $2\pi r$, $r > 0$. Whether M is isometric to a euclidean n-sphere S_r^n of radius r is usually referred to as the *Blaschke problem* (for spheres).

Recently, Berger [1] made use of an inequality given by Kazdan [3] to prove that

$$\operatorname{vol} M \geqslant \operatorname{vol} S_r^n,$$

and that the equality holds iff M is isometric to S_r^n. On the other hand, Weinstein [4] has proved the following result. If M is a connected compact Riemannian n-manifold in which all geodesics are smoothly closed and have the same length, say $2\pi r$, if UM is the space of unit tangent vectors of M, CM is the space of (oriented) closed geodesics in M, α is the Euler class of the natural circle fibration π: $UM \to CM$, and CM is so oriented that the value $\langle \alpha^{n-1}, [CM] \rangle$ of α^{n-1} at the fundamental class $[CM]$ is positive, then

$$2 \operatorname{vol} M = \langle \alpha^{n-1}, [CM] \rangle \operatorname{vol} S_r^n.$$

Therefore the evaluation of vol M depends only on that of $\langle \alpha^{n-1}, [CM] \rangle$. It is remarked in [4] that, when n is even, $\langle \alpha^{n-1}, [CM] \rangle = 2$. Hence for any even $n \geqslant 2$, $\operatorname{vol} M = \operatorname{vol} S_r^n$, and thus M and S_r^n are isometric.

The purpose of this paper is to show that for any odd $n > 1$, $\langle \alpha^{n-1}, [CM] \rangle = 2$ remains valid and hence M and S_r^n are isometric. The Blaschke problem (for spheres) is thus completely solved.

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Throughout this paper, M denotes a connected compact Riemannian n-manifold (without boundary), $n > 1$, in which all geodesics are smoothly closed and have the same length, UM denotes the space of unit tangent vectors of M, and CM denotes the space of (oriented) closed geodesics in M. It is clear that UM is a smooth $(2n-1)$-manifold, CM is a smooth $(2n-2)$-manifold, and there are a natural smooth $(n-1)$-sphere fibration $p\colon UM \to M$ and a natural smooth circle fibration $\pi\colon UM \to CM$ such that for any $v \in UM$, v is the unit tangent vector of πv at pv.

Lemma 1. *Assume that M has the integral cohomology groups of the n-sphere. Then the integral cohomology groups of UM and CM are given as follows. If n is even ($\geqslant 2$), then*

$$H^k(UM) = \begin{cases} Z & \text{for } k = 0, 2n-1, \\ Z_2 & \text{for } k = n, \\ 0 & \text{otherwise}; \end{cases}$$

$$H^k(CM) = \begin{cases} Z & \text{for } k = 0, 2, 4, \cdots, 2n-2, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, the homomorphism $H^{k-2}(CM) \to H^k(CM)$, appearing in the Gysin sequence of $\pi\colon UM \to CM$, is an isomorphism for $k = 0, 2, \cdots, n-2, n+2, \cdots, 2n-2$, and is a monomorphism of cokernel Z_2 for $k = n$. If n is odd (> 1), then

$$H^k(UM) = \begin{cases} Z & \text{for } k = 0, n-1, n, 2n-1, \\ 0 & \text{otherwise}; \end{cases}$$

$$H^k(CM) = \begin{cases} Z & \text{for } k = 0, 2, 4, \cdots, n-3, n+1, \cdots, 2n-2, \\ Z \oplus Z & \text{for } k = n-1, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, there are exact sequences

$$0 \to H^{n-3}(CM) \to H^{n-1}(CM) \to H^{n-1}(UM) \to 0,$$

$$0 \to H^n(UM) \to H^{n-1}(CM) \to H^{n+1}(CM) \to 0,$$

which are parts of the Gysin sequence of $\pi\colon UM \to CM$.

Proof. The result is well-known and is included here for the sake of completeness and reference.

Since M has the integral cohomology groups of the n-sphere it is orientable. Therefore the Gysin sequence of $p\colon UM \to M$, i.e.,

$$\cdots \to H^{k-n}(M) \overset{\cup \alpha(p)}{\to} H^k(M) \overset{p^*}{\to} H^k(UM) \to H^{k-n+1}(M) \to \cdots$$

is exact, where $\alpha(p)$ is the Euler class for p: $UM \to M$. We know that $\alpha(p)$ is equal to 0 or the double of the fundamental class of M according as n is odd or even. Hence it is easy to compute $H^k(UM)$ as asserted.

From the homotopy sequence of π: $UM \to CM$, it is seen that π_*: $\pi_1(UM) \to \pi_1(CM)$ is surjective. Therefore, by Hurewicz's theorem, π_*: $H_1(UM) \to H_1(CM)$ is surjective. Hence $H_1(CM) = 0$ and consequently CM is orientable. Because of this fact, the Gysin sequence of π: $UM \to CM$, i.e.,

$$\cdots \to H^{k-2}(CM) \overset{\cup\alpha}{\to} H^k(CM) \overset{\pi^*}{\to} H^k(UM) \to H^{k-1}(CM) \to \cdots$$

is exact, where α is the Euler class for π: $UM \to CM$. Now it is easy to compute $H^k(CM)$ and to verify asserted properties of $H^k(CM)$.

As an immediate consequences of Lemma 1, we have

Lemma 2. *For any even $n \geqslant 2$, if M has the integral cohomology groups of the n-sphere, then $\langle \alpha^{n-1}, [CM] \rangle = 2$.*

Now we are in a position to examine whether Lemma 2 remains valid for any odd $n > 2$. Hereafter, we let $n = 2m + 1$, where m is an integer $\geqslant 1$. Also we assume that M has the following properties. First, M has the integral cohomology groups of the $(2m + 1)$-sphere. Secondly, there is a point y of M such that any closed geodesic in M does not have y as a point of self-intersection. Notice that the second property is clearly satisfied by any wiedersehen manifold.

It is easily seen from Lemma 1 that for any $k = 1, \cdots, m - 1$, α^k is a generator of $H^{2k}(CM)$, and that if b is an element of $H^{2m}(CM)$ such that $\pi^* b$ is a generator of $H^{2m}(UM)$, then $\{b, \alpha^m\}$ is a basis of $H^{2m}(CM)$. In the following, we shall find a specified b which enables us to compute $\langle \alpha^{2m}, [CM] \rangle$.

Lemma 3. *Let a be a generator of the image of $H^{2m+1}(UM) \to H^{2m}(CM)$ (see Lemma 1). Then*

$$a \cup a = 2g$$

for some generator g of $H^{4m}(CM)$.

Instead of proving Lemma 3, we prove its dual which is given in terms of integral homology groups as follows.

Lemma 3'. *Let a^* be a generator of the image of π_*: $H_{2m}(UM) \to H_{2m}(CM)$. Then CM can be so oriented that $a^* \cap a^* = 2$.*

Proof. By hypothesis, there is a point y of M such that any closed geodesic in M does not have y as a point of self-intersection. Such a point y has a neighborhood V such that for any $v \in p^{-1}y$, $p\pi^{-1}\pi v \cap V$ is a single open arc containing y. Then it is easily seen that $\pi^{-1}\pi p^{-1}y \cap p^{-1}(V - \{y\})$

contains exactly two components, each of which is mapped homeomorphically onto $V - \{y\}$ by p. Notice that if C is one of the components, then the other component if $\{-v | v \in C\}$.

Let z be a point of V different from y, and let γ be an oriented closed geodesic in M passing through both y and z. Then

$$\pi p^{-1}y \cap \pi p^{-1}z = \{\gamma, -\gamma\}.$$

Let $p^{-1}y$ and $p^{-1}z$ be oriented so that they represent the same generator of $H_{2m}(UM)$. Then we may let $\pi p^{-1}y$ and $\pi p^{-1}z$ be $2m$-cycles representing a^*. Therefore we have only to show that CM can be so oriented that the intersection number of $\pi p^{-1}y$ and $\pi p^{-1}z$ is equal to 1 at both γ and $-\gamma$.

Consider the $2m$-sphere bundle

$$p: p^{-1}(V - \{y\}) \to V - \{y\}.$$

Since $p^{-1}z$ is a fibre of the $2m$-sphere bundle and since each of the two components of $\pi^{-1}\pi p^{-1}y \cap p^{-1}(V - \{y\})$ is a cross-section, it follows that $\pi^{-1}\pi p^{-1}y$ and $p^{-1}z$ intersect at exactly two points, and the intersection number at either point is equal to 1 or -1. Hence the intersection number of $\pi p^{-1}y$ and $\pi p^{-1}z$ at each of γ and $-\gamma$ is equal to 1 or -1.

Let

$$\lambda : UM \to UM, \quad \lambda' : CM \to CM$$

be the involutions defined by

$$\lambda(v) = -v, \quad \lambda'(\xi) = -\xi.$$

Then

$$\begin{array}{ccccc} M & \overset{p}{\longleftarrow} & UM & \overset{\pi}{\longrightarrow} & CM \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \lambda} & & \uparrow{\scriptstyle \lambda'} \\ M & \overset{p}{\longleftarrow} & UM & \overset{\pi}{\longrightarrow} & CM \end{array}$$

is commutative. Since M is odd-dimensional, λ is orientation-reversng so that λ' is orientation-preserving. Therefore the intersection number of $\pi p^{-1}y$ and $\pi p^{-1}z$ at $-\gamma = \lambda'\gamma$ is equal to that of $\lambda'\pi p^{-1}y$ and $\lambda'\pi p^{-1}z$ at γ and thus is equal to that of $\pi p^{-1}y$ and $\pi p^{-1}z$ at γ. Hence the proof is complete.

Lemma 4. *There is a basis* $\{b, \alpha^m\}$ *of* $H^{2m}(CM)$ *such that if* a *and* g *are as in Lemma* 3, *then*

(i) $a \cup b = g$,

(ii) $a = 2b - \alpha^m$.

Proof. Since the exact sequences

$$0 \to H^{2m-2}(CM) \to H^{2m}(CM) \to H^{2m}(UM) \to 0,$$
$$0 \leftarrow H^{2m+2}(CM) \leftarrow H^{2m}(CM) \leftarrow H^{2m+1}(UM) \leftarrow 0$$

are dual to each other, there is an element b of $H^{2m}(CM)$ such that

$$a \cup b = g,$$

and $\{b, \alpha^m\}$ is a basis of $H^{2m}(CM)$.

Let

$$a = \beta b + \gamma \alpha^m,$$

where β and γ are integers. We know from Lemma 3 that

$$a \cup a = 2g, \quad a \cup \alpha = 0.$$

Therefore

$$2g = a \cup (\beta b + \gamma \alpha^m) = \beta g,$$

so that $\beta = 2$. Hence

$$a = 2b + \gamma \alpha^m.$$

Since

$$g = a \cup b = (2b + \gamma\alpha^m) \cup b = 2(b \cup b) + \gamma(\alpha^m \cup b),$$

it follows that γ is odd, say $\gamma = 2k - 1$. Let

$$b' = b + k\alpha^m.$$

Then $\{b', \alpha^m\}$ is a basis of $H^{2m}(CM)$ such that $a \cup b' = g$ and $a = 2b' - \alpha^m$. Hence our assertion follows by using b' in place of b.

Lemma 5. $\langle \alpha^{2m}, [CM] \rangle = 2.$

Proof. Let $\{b, \alpha^m\}$ be the basis of $H^{2m}(CM)$ given in Lemma 4. Then

$$b \cup b = rg$$

for some integer r. Since

$$b \cup \alpha^m = b \cup (2b - a) = (2r - 1)g,$$

$$\alpha^m \cup \alpha^m = (2b - a) \cup (2b - a) = (4r - 2)g,$$

it follows from Poincaré duality that

$$\pm 1 = \begin{vmatrix} \langle b \cup b, [CM] \rangle & \langle b \cup \alpha^m, [CM] \rangle \\ \langle \alpha^m \cup b, [CM] \rangle & \langle \alpha^m \cup \alpha^m, [CM] \rangle \end{vmatrix}$$

$$= \begin{vmatrix} r & 2r - 1 \\ 2r - 1 & 4r - 2 \end{vmatrix} = 2r - 1.$$

Therefore $r = 0$ or 1 so that $\langle e^{2m}, [CM] \rangle = \pm 2$. Since CM is so oriented that $\langle \alpha^{2m}, [CM] \rangle$ is positive, our assertion follows.

Combining Lemmas 2 and 5 and Weinstein's theorem [4], we have

Theorem 1. *Let M be a connected compact Riemannian n-manifold without boundary, $n \geqslant 2$, which has the integral cohomology groups of the n-sphere and in which all geodesics are smoothly closed and have the same length, say $2\pi r$. If n is odd, it is also assumed that there is a point of M which is not a point of* *self-intersection of any closed geodesic in M. Then the volume of M is equal to that of a euclidean n-sphere of radius r.*

Since wiedersehen n-manifolds satisfy the hypothesis of Theorem 1, Theorem 1 and results of Berger [1] and Kazdan [3] yield

Theorem 2. *Any wiedersehen n-manifold is isometric to a euclidean sphere.*

References

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